II Semester M.Sc. Degree Examination, June/July 2014
(RNS) (2011-12 & Onwards)
MATHEMATICS
M-201 : Algebra – II

Time : 3 Hours Max. Marks : 80

Instructions : *i) Answer **any five (5) full** questions choosing atleast **two** from **each** Part.*

*ii) **All** questions carry **equal** marks.*

PART – A

1. a) Define the degree of an extension K of a field F. If L is a finite extension of K and K is a finite extension of F, then prove that L is a finite extension of F. Moreover, prove that [L : F] = [L : K] [K : F]. **6**

   b) Let K be an extension of a field F and a, b $\in$ K be algebraic over F of degree m and n respectively. If m and n are relatively prime, prove that F(a, b) is of degree mn over F. **6**

   c) Show that $Q\left(\sqrt{2}+\sqrt{3}\right)$ is an algebraic extension of Q of degree 4. **4**

2. a) Prove that a polynomial of degree n over a field F can have atmost n roots in any extension field K. Is the result true when K is not a field ? Explain. **8**

   b) Define a splitting field of a polynomial $f(x) \in F[x]$. Prove that any two splitting fields E and E′ of the polynomials $f(x) \in F[x]$ and $f'(t) \in F'[t]$, respectively are isomorphic by an isomorphism ϕ with the property that $\alpha\phi = \alpha Z = \alpha'$ for $\alpha \in F$, where $Z : F \to F'$ is an isomorphism. Hence, deduce that any two splitting fields of the same polynomial over a given field F are isomorphic by an isomorphism leaving every element of F fixed. **8**

3. a) Prove or disprove : "A regular septagon is constructible". **6**

b) Prove that any finite extension of a field F of characteristic zero, is a simple extension. **4**

c) If the number α satisfies an irreducible polynomial of degree k, over the field of rationals and k is not a power of 2, then show that α is not a constructible number. **6**

4. a) If K is finite extension of F, then prove that G(K, F) is a finite group and its order $o(G(K,F))$ satisfies the condition $o(G(K,F)) \le [K:F]$. **6**

b) Let K be a finite Galois extension of a field F and let T be an intermediate field of K and F. Prove that (i) T is a normal extension of F if and only if G(K, T) is a normal subgroup of G(K, F). (ii) When T is a normal extension of F, then G(T, F) is isomorphic to G(K, F)/G(K, T). **10**

PART – B

5. a) If V is an n-dimensional vector space over F, then show that for a given T in A(V) there exists a non-trivial polynomial $q(x) \in F[x]$ of degree at most n^2, such that q(T) = 0. **6**

b) If V is a finite dimensional vector space over F and if $T \in A(V)$ is right invertible, then show that T is invertible. **4**

c) Define the range and rank of a linear transformation T. If V is finite dimensional vector space over F, then show that $T \in A(V)$ is regular if and only if T maps V onto V. **6**

6. a) Let $T \in A(V)$. Prove that the non-zero characteristic vectors belonging to distinct characteristic roots are linearly independent. **6**

b) Let V be the set of all polynomials in x of degree 3 or less over F. On V, let T be the transformation given by $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)T = \alpha_0 + \alpha_1(1+x) + \alpha_2(1+x)^2 + \alpha_3(1+x)^3$.

Compute the matrix of T in the basis

i) $1, x, x^2, x^3$

ii) $1, 1+x, 1+x^2, 1+x^3$. **4**

c) If V is n-dimensional over F and if $T \in A(V)$ has matrix $M_1(T)$ in the basis $v_1, v_2, ..., v_n$ and the matrix $M_2(T)$ in the basis $w_1, w_2, ..., w_n$ of V over F, then prove that there exists $S \in A(V)$ defined as $v_iS = w_i$, $1 < i < n$; such that $M_2(T) = M(S) \cdot M_1(T) \cdot (M(S))^{-1}$. **6**

7. a) If the matrix $A \in F_n$ has all its characteristic roots in F, then show that there is a matrix $C \in F_n$ such that CAC^{-1} is a triangular matrix. **6**

b) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F. **5**

c) Define a nilpotent transformation. Let $T \in A(V)$ and V_1 be an n_1-dimensional subspace of an n-dimensional vector space V spanned by $\{v, vT, ..., vT^{n_1-1}\}$, where $v \neq 0$. If $u \in V_1$ is such that $uT^{n_1-k} = 0$, $0 < k \leq n_1$, then show that $u = u_0T^k$ for some $u_0 \in V_1$. **5**

8. a) Define the Jordan Cannonical form of a matrix. Find all possible Jordan forms for all 8 × 8 matrices having $x^2(x-1)^3$ as minimal polynomial. **6**

b) Define a normal transformation. If λ is a characteristic root of the normal transformation N and if $vN = \lambda v$ then show that $vN^* = \bar{\lambda}v$. **6**

c) Define rank, signature and real quadratic form. Determine the rank and signature of the following real quadratic form $x_1^2 + 2x_1x_2 + x_2^2$. **4**